

ABSOLUTE TORSION

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Dedicated to Mel Rothenberg on the occasion of his 65-th birthday

ABSTRACT. In this paper we use the results of our previous work [FT] in order to compute *the phase of the torsion* of an Euler structure ξ in terms of the characteristic class $c(\xi)$. Also, we introduce here a new notion of *an absolute torsion*, which does not require a choice of any additional topological information (like an Euler structure). We prove that in the case of closed 3-manifolds obtained by 0-surgery on a knot in S^3 the absolute torsion is equivalent to the Conway polynomial. Hence the absolute torsion can be viewed as a high-dimensional generalization of the Conway polynomial.

§1. Euler structures and Poincaré-Reidemeister metric

In this section we give a brief review of the main results of [FT], which we will use in the sequel. The proofs of all theorems, appearing in this section, can be found in [FT].

1.1. Determinant lines. We shall denote by \mathbf{k} a fixed ground field of characteristic zero. The most important special cases are $\mathbf{k} = \mathbf{R}$ and $\mathbf{k} = \mathbf{C}$.

If V is a finite dimensional vector space over \mathbf{k} , *the determinant line of V* is denoted by $\det V$ and is defined as the top exterior power of V , i.e., $\Lambda^n V$, where $n = \dim V$. The dual line $\text{Hom}_{\mathbf{k}}(V, \mathbf{k})$ is denoted by $(\det V)^{-1}$. For a finite dimensional graded vector space $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$, its *determinant line* $\det V$ is defined as the tensor product $\det V = \det V_0 \otimes (\det V_1)^{-1} \otimes \det V_2 \otimes \cdots \otimes (\det V_m)^{(-1)^m}$.

Let C be a finite dimensional chain complex over \mathbf{k} . In the theory of torsion a crucial role is played by a canonical isomorphism

$$\varphi_C : \det C \rightarrow \det H_*(C), \quad (1-1)$$

where both C and $H_*(C)$ are considered as graded vector spaces. The definition of the mapping φ_C is as follows. Choose for each $q = 0, \dots, m$ non-zero elements $c_q \in \det C_q$ and $h_q \in \det H_q(C)$. Set $c = c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m} \in \det C$ and $h = h_0 \otimes h_1^{-1} \otimes h_2 \otimes \cdots \otimes h_m^{(-1)^m} \in \det H_*(C)$, where -1 in the exponent denotes the dual functional.

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We define φ_C by $\varphi_C(c) = (-1)^{N(C)} [c : h] h$, where $N(C)$ is a residue modulo 2 defined below and $[c : h]$ is a nonzero element of \mathbf{k} , defined by

$$[c : h] = \prod_{q=0}^m [d(b_{q+1}) \hat{h}_q b_q / \hat{c}_q]^{(-1)^{q+1}}. \quad (1-2)$$

Here b_q is a sequence of vectors of C_q whose image $d(b_q)$ under the boundary homomorphism $d : C_q \rightarrow C_{q-1}$ is a basis of $\text{Im } d$; the symbol \hat{h}_q denotes a sequence of cycles in C_q such that the wedge product of their homology classes equals h_q ; the symbol \hat{c}_q denotes a basis of C_q whose wedge product equals c_q ; the number $[d(b_{q+1}) \hat{h}_q b_q / \hat{c}_q]$ is the determinant of the matrix transforming \hat{c}_q into the basis $d(b_{q+1}) \hat{h}_q b_q$ of C_q . The residue $N(C)$ is defined by

$$N(C) = \sum_{q=0}^m \alpha_q(C) \beta_q(C) \pmod{2}, \quad (1-3)$$

where

$$\alpha_q(C) = \sum_{j=0}^q \dim C_j \pmod{2}, \quad \beta_q(C) = \sum_{j=0}^q \dim H_j(C) \pmod{2}. \quad (1-4)$$

Formula (1-2) involves the sign refinement of the standard formula, introduced in [T1]. We will introduce now more sign involving factors in other natural maps arising in this setting.

1.2. The fusion isomorphism. For two finite-dimensional graded vector spaces $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ and $W = W_0 \oplus W_1 \oplus \cdots \oplus W_m$, we define a canonical isomorphism

$$\mu_{V,W} : \det V \otimes \det W \rightarrow \det(V \oplus W), \quad (1-5)$$

by

$$\mu_{V,W} = (-1)^{M(V,W)} \bigotimes_q \mu_q^{(-1)^q}, \quad (1-6)$$

where $\mu_q : \det V_q \otimes \det W_q \rightarrow \det(V_q \oplus W_q)$ is the isomorphism defined by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge w_2 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_l \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_k,$$

with $k = \dim V_q, l = \dim W_q$, the isomorphism μ_q^{-1} is defined as the transpose of the inverse of μ_q , and

$$M(V, W) = \sum_{q=1}^m \alpha_{q-1}(V) \alpha_q(W) \in \mathbf{Z}/2\mathbf{Z}, \quad (1-7)$$

with $\alpha_{q-1}(V)$ and $\alpha_q(W)$ defined as in (1-4).

We will call (1-5) *the fusion homomorphism*.

1.3. Duality operator. Let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ be a finite dimensional graded vector space over \mathbf{k} with odd m . We define the *dual graded vector space* over \mathbf{k} by $V' = V'_0 \oplus V'_1 \oplus \cdots \oplus V'_m$ where $V'_q = (V_{m-q})^* = \text{Hom}_{\mathbf{k}}(V_{m-q}, \mathbf{k})$. We define a *duality operator*

$$D = D_V : \det V \rightarrow \det V'$$

as follows. Let $v_q \in \det V_q$ be a volume element determined by a basis of V_q and let $v'_{m-q} \in \det V'_{m-q}$ be the volume element determined by the dual basis of V'_{m-q} , for $q = 0, 1, \dots, m$. Then

$$D(v_0 \otimes v_1^{-1} \otimes v_2 \otimes \cdots \otimes v_m^{-1}) = (-1)^{s(V)} v'_0 \otimes (v'_1)^{-1} \otimes v'_2 \otimes \cdots \otimes (v'_m)^{-1},$$

where the residue $s(V) \in \mathbf{Z}/2\mathbf{Z}$ is given by

$$s(V) = \sum_{q=1}^m \alpha_{q-1}(V) \alpha_q(V) + \sum_{q=0}^{(m-1)/2} \alpha_{2q}(V). \quad (1-8)$$

1.4. Euler structures and their characteristic classes. We recall the notion of combinatorial Euler structure on a CW-space, following [T2].

Let X be a finite connected CW-space with $\chi(X) = 0$. An *Euler chain in X* is a singular 1-chain ξ in X such that $d\xi = \sum_a (-1)^{|a|} p_a$ where a runs over all cells of X and p_a is a point in a ; the symbol $|a|$ denotes the dimension of a .

An *Euler structure on X* is an equivalence class of Euler chains with respect to a natural equivalence relation. The set of equivalence classes (i.e., the set of Euler structures on X) is denoted by $\text{Eul}(X)$. We shall usually denote an Euler structure and a representing it Euler chain by the same letter.

The group $H_1(X)$ acts on the set $\text{Eul}(X)$ and this action is free and transitive. We shall use multiplicative notation both for this action and for the group operation in $H_1(X)$.

Assume that X is a closed connected PL-manifold with $\chi(X) = 0$. For each Euler structure ξ on X we define its *characteristic class* $c(\xi) \in H_1(X)$ as follows. Choose a PL-triangulation ρ of X . Let W be the 1-chain in X defined by $W = \sum_{a_0 < a_1 \in \rho} (-1)^{|a_0|+|a_1|} \langle \underline{a}_0, \underline{a}_1 \rangle$, where a_1 runs over all simplices of ρ , a_0 runs over all proper faces of a_1 , and $\langle \underline{a}_0, \underline{a}_1 \rangle$ is a path in a_1 going from the barycenter \underline{a}_0 of a_0 to the barycenter \underline{a}_1 of a_1 . It is easy to check (see [HT]) that $\partial W = (1 - (-1)^m) \sum_{a \in \rho} (-1)^{|a|} \underline{a}$ where $m = \dim X$. Now, any Euler structure on X can be presented by an Euler chain ξ in (X, ρ) such that $\partial \xi = \sum_a (-1)^{|a|} \underline{a}$. It is clear that $(1 - (-1)^m) \xi - W$ is a 1-cycle. Denote its class in $H_1(X)$ by $c(\xi)$. In this way, we obtain a mapping $c : \text{Eul}(X) \rightarrow H_1(X)$.

If m is odd, then (in multiplicative notation) $c(h\xi) = h^2 c(\xi)$ for any $\xi \in \text{Eul}(X)$, $h \in H_1(X)$. For odd m , the mod 2 reduction of $c(\xi)$ is independent of ξ and equals to the dual of the Stiefel-Whitney class $w_{m-1}(X) \in H^{m-1}(X, \mathbf{Z}/2\mathbf{Z})$. This follows from the fact that $W \pmod{2}$ represents the dual of $w_{m-1}(X)$, see [HT].

1.5. Torsion of Euler structures. Let F be a flat \mathbf{k} -vector bundle over a finite connected CW-space X . For each Euler structure $\xi \in \text{Eul}(X)$ on X we define a torsion $\tau(X, \xi; F)$ which is an element of the determinant line $\det H_*(X; F)$ defined

up to multiplication by $(-1)^{\dim F}$. We denote by $C_*(X; F)$ the cellular chain complex computing the homology of X with values in F . Recall that

$$C_q(X; F) = \bigoplus_{\dim a=q} \Gamma(a, F), \quad (1-9)$$

where $\Gamma(a, F)$ denotes the space of flat sections of F over a . Set

$$\tau(X, \xi; F) = \varphi_C(c_0 \otimes c_1^{-1} \otimes c_2 \otimes \cdots \otimes c_m^{(-1)^m}) \in \det H_*(X; F) \quad (1-10)$$

where $m = \dim X$ and $c_q \in \det C_q(X; F)$ ($q = 0, 1, \dots, m$) are non-zero elements defined as follows. Fix a point $x \in X$ and a basis e_x in the fiber F_x . Let $\beta_a : [0, 1] \rightarrow X$ be a path connecting $x = \beta_a(0)$ to a point $\beta_a(1) \in a$. The assumption $\chi(X) = 0$ implies that the 1-chain $\sum_a (-1)^{|a|} \beta_a$ (where a runs over all cells of X) is an Euler chain with boundary $\sum_a (-1)^{|a|} \beta_a(1)$. We choose the paths $\{\beta_a\}_a$ so that this chain represents ξ . We apply the parallel transport to e_x along β_a to obtain a basis in the fiber $F_{\beta_a(1)}$ and we extend it to a basis of flat sections over a . The concatenation of these bases over all q -dimensional cells gives a basis in $C_q(X; F)$ via (1-9). The wedge product of the elements of this basis yields $c_q \in \det C_q(X; F)$.

1.6. Lemma. *If $\dim F$ is even then the torsion $\tau(X, \xi; F) \in \det H_*(X; F)$ is well defined, has no indeterminacy and is combinatorially invariant. If $\dim F$ odd, the torsion $\tau(X, \xi; F)$ has a sign indeterminacy.*

In order to fix the sign of the torsion in the case $\dim F$ odd, one may use technique of *homological orientations*, i.e., the orientations of the determinant line $\det H_*(X, \mathbf{R})$ of the real cohomology, which was introduced in [T1, T2]. Cf. also 3.3.

1.7. The Poincaré-Reidemeister scalar product. Let X be a *closed connected oriented piecewise linear manifold of odd dimension m* . Let F be a flat \mathbf{k} -vector bundle over X . The standard homological intersection pairing

$$H_q(X; F^*) \otimes H_{m-q}(X; F) \rightarrow \mathbf{k} \quad (1-11)$$

allows us to identify the dual of $H_{m-q}(X; F)$ with $H_q(X; F^*)$. Applying the construction of Section 1.3 to the graded vector space $\bigoplus_{q=0}^m H_q(X; F)$ we obtain a canonical isomorphism

$$D : \det H_*(X; F) \rightarrow \det H_*(X; F^*). \quad (1-12)$$

It is easy to check that D does not depend on the choice of the orientation of X .

The *Poincaré-Reidemeister scalar product* is defined as the bilinear pairing

$$\langle \cdot, \cdot \rangle_{PR} : \det H_*(X; F) \times \det H_*(X; F) \rightarrow \mathbf{k}, \quad (1-13)$$

given by

$$\langle a, b \rangle_{PR} = \mu(a \otimes D(b)) / \tau(X; F \oplus F^*) \in \mathbf{k},$$

where $a, b \in \det H_*(X; F)$ and D is the isomorphism (1-12). Here μ denotes the canonical fusion isomorphism

$$\det H_*(X; F) \otimes \det H_*(X; F^*) \rightarrow \det (H_*(X; F) \oplus H_*(X; F^*)) = \det H_*(X; F \oplus F^*)$$

defined in Section 1.2.

The Poincaré-Reidemeister scalar product determines the *Poincaré-Reidemeister metric* (or norm) on the determinant line $\det H_*(X; F)$, which was introduced in [Fa]. It is given by

$$a \mapsto \sqrt{|\langle a, a \rangle_{PR}|}, \quad a \in \det H_*(X; F)$$

(the positive square root of the absolute value of $\langle a, a \rangle_{PR}$). The PR-scalar product contains an additional phase or sign information.

The following Theorem computes the PR-scalar product in terms of the Euler structures.

1.8. Theorem. *Let F be a flat \mathbf{k} -vector bundle over a closed connected orientable PL-manifold X of odd dimension m . If $\dim F$ is odd, then we additionally assume that X is provided with a homology orientation. Then for any $\xi \in \text{Eul}(X)$,*

$$\langle \tau(X, \xi; F), \tau(X, \xi; F) \rangle_{PR} = (-1)^z \det_F(c(\xi)), \quad (1-14)$$

where $\langle \cdot, \cdot \rangle_{PR}$ is the Poincaré-Reidemeister scalar product, $\det_F(c(\xi)) \in \mathbf{k}^*$ is the determinant of the monodromy of F along the characteristic class $c(\xi) \in H_1(X)$, and the residue $z \in \mathbf{Z}/2\mathbf{Z}$ is

$$z = \begin{cases} 0, & \text{if } \dim F \text{ is even or } m \equiv 3 \pmod{4}, \\ s\chi(X) \pmod{2}, & \text{if } \dim F \text{ is odd and } m \equiv 1 \pmod{4}, \end{cases} \quad (1-15)$$

where $s\chi(X) = \sum_{i=0}^{(m-1)/2} \dim H^{2i}(X; \mathbf{R})$ denotes the semi-characteristic of X .

The following theorem describes the sign of the PR-pairing.

1.9. Theorem. *Let F be a flat \mathbf{R} -vector bundle over a closed connected orientable PL-manifold X of odd dimension m . The Poincaré-Reidemeister scalar product on $\det H_*(X; F)$ is positive definite for $m \equiv 3 \pmod{4}$. If $m \equiv 1 \pmod{4}$ then the sign of the Poincaré-Reidemeister scalar product equals*

$$(-1)^{\langle w_1(F) \cup w_{m-1}(X), [X] \rangle + s\chi(X) \cdot \dim F}. \quad (1-16)$$

Note two interesting special cases of Theorem 1.9 assuming $m \equiv 1 \pmod{4}$. If $w_{m-1}(X) = 0$ and $\dim F$ is even then the Poincaré-Reidemeister scalar product is positive definite. The same conclusion holds if F is orientable and $\dim F$ is even.

1.10. Theorem (Analytic torsion and Euler structures). *Let X be a closed connected orientable smooth manifold of odd dimension and let F be a flat \mathbf{R} -vector bundle over X . If $\dim F$ is odd, then we additionally assume that X is provided with a homology orientation. For any Euler structure $\xi \in \text{Eul}(X)$, the Ray-Singer norm of its cohomological torsion $\tau^\bullet(X, \xi; F) \in \det H^*(X; F)$ (defined similarly to 1.5) is equal to the positive square root of the absolute value of the monodromy of F along the characteristic class $c(\xi) \in H_1(X)$:*

$$\|\tau^\bullet(X, \xi; F)\|^{RS} = |\det_F c(\xi)|^{1/2}. \quad (1-17)$$

In the special case, where the flat bundle F is acyclic, i.e., $H^*(X; F) = 0$, the torsion $\tau^\bullet(X, \xi; F)$ is a real number and Theorem 1.10 yields

$$\prod_{q=0}^{\dim X} (\mathfrak{Det} \Delta'_q)^{(-1)^{q+1} q} = \frac{(\tau^\bullet(X, \xi; F))^2}{|\det_F c(\xi)|}. \quad (1-18)$$

Theorem 1.10 generalizes both the classical Cheeger-Müller theorem [C], [Mu1] (dealing with orthogonal flat real bundles F) and also the (more general) theorem of Müller [Mu2] (dealing with the unimodular flat real bundles F). Note that if F is unimodular then $|\det_F c(\xi)| = 1$ and the torsion $\tau^\bullet(X, \xi; F)$ does not depend on the choice of ξ .

§2. Phase of the torsion

The purpose of this section is to give a formula expressing the phase of the torsion of Euler structures $\xi \in \text{Eul}(X)$ (understood as an element of a determinant line) in terms of the characteristic class $c(\xi) \in H_1(X)$.

Throughout this section $\mathbf{k} = \mathbf{C}$.

2.1. Involution on the determinant line. Let X be a closed orientable piecewise linear manifold of odd dimension m . Let $F \rightarrow X$ be a flat complex vector bundle admitting a flat Hermitian metric.

We introduce a canonical involution on the complex line $\det H_*(X; F)$. Consider the duality operator (1-12). Recall that F^* denotes the dual flat vector bundle. Any flat Hermitian metric on F determines an anti-linear isomorphism of flat bundles $F^* \rightarrow F$, which induces an anti-linear isomorphism

$$\psi : \det H_*(X; F^*) \rightarrow \det H_*(X; F). \quad (2-1)$$

Definition. *The canonical involution on the determinant line $\det H_*(X; F)$ is the following anti-linear isomorphism*

$$- : \det H_*(X; F) \rightarrow \det H_*(X; F), \quad (2-2)$$

where for $\tau \in \det H_*(X; F)$ we set

$$\bar{\tau} = (-1)^{s\chi(X) \cdot \dim F \cdot (m+1)/2} \psi(D(\tau)), \quad (2-3)$$

where $m = \dim X$.

Here $s\chi(X)$ denotes the semicharacteristic of X , i.e. $s\chi(X) = \sum_{i=0}^{(m-1)/2} b_{2i}(X)$.

2.2. Lemma. (A) *The anti-linear isomorphism (2-2) is an involution.*

(B) *It is independent of the choice of a Hermitian metric on F .*

(C) *If the flat bundle F is acyclic then the determinant line $\det H_*(X; F)$ is canonically isomorphic to \mathbf{C} and under this isomorphism the involution (2-2) coincides with the complex conjugation.*

A proof is given below.

An element $\tau \in \det H_*(X; F)$ will be called *real* if $\bar{\tau} = \tau$. The real elements of $\det H_*(X; F)$ form a real line.

For a nonzero $\tau \in \det H_*(X; F)$, its *phase* is defined as angle $\phi \in \mathbf{R}$ so that τ can be represented in the form $\tau = \tau_0 e^{i\phi}$, where τ_0 is real. It is clear that the phase ϕ is determined up to adding integral multiples of π . We will denote it by $\mathbf{Ph}(\tau)$.

The following theorem computes the phase of the torsion in terms of the characteristic class of the Euler structure.

2.3. Theorem. *Let X be a closed orientable piecewise linear manifold of odd dimension m and let $F \rightarrow X$ be a flat complex vector bundle admitting a flat Hermitian metric. Then the phase of the torsion $\tau(\xi, F)$ is given by the following formula:*

$$\mathbf{Ph}(\tau(X, \xi; F)) = \frac{1}{2} \arg \det_F(c(\xi)) \mod \pi \mathbf{Z}. \quad (2-4)$$

Note that in the case when $\dim F$ is odd, the torsion $\tau(X, \xi; F)$ is defined only up to a sign, but its phase is still well defined.

The number $d = \det_F(c(\xi))$ lies on the unit circle and $\arg(d) = \arg \det_F(c(\xi)) \in \mathbf{R}/(2\pi \mathbf{Z})$ is defined by $d = \exp(i \arg(d))$.

Proof of Lemma 2.2 and Theorem 2.3. Let τ denote $\tau(X, \xi; F)$ (in the case when $\dim F$ is even) or $\tau(X, \eta, \xi; F)$ (in the case when $\dim F$ is odd); here $\xi \in \text{Eul}(X)$ is an Euler structure and η is a homological orientation (i.e. an orientation of the line $\det H_*(X; \mathbf{R})$) which we need in order to fix the sign of the torsion in the case when $\dim F$ is odd, cf. 6.3 of [FT].

We first show that

$$\bar{\tau} = \det_F(c(\xi))^{-1} \cdot \tau. \quad (2-5)$$

We will use theorem 7.2 of [FT], which states

$$D(\tau) = (-1)^{\dim F \cdot s\chi(X) \cdot (m+1)/2} \cdot \tau(X, \eta, \xi^*; F^*), \quad (2-6)$$

where $\xi^* \in \text{Eul}(X)$ is the dual Euler structure. Since $\xi = c(\xi)\xi^*$ (cf. formula (5-4) in [FT]) we obtain

$$\begin{aligned} D(\tau) &= (-1)^{\dim F \cdot s\chi(X) \cdot (m+1)/2} \cdot \det_{F^*}(c(\xi))^{-1} \cdot \tau(X, \eta, \xi; F^*) = \\ &= (-1)^{\dim F \cdot s\chi(X) \cdot (m+1)/2} \cdot \det_F(c(\xi)) \cdot \tau(X, \eta, \xi; F^*), \end{aligned} \quad (2-7)$$

where $m = \dim X$. Applying to both sides of (2-7) the anti-linear isomorphism (2-1) and using our definition (2-3) we obtain (2-5).

To prove statement (A) of Lemma 2.2, we note that according to (2-5) we have

$$\bar{\tau} = e^{i\psi} \tau, \quad \tau \in \det H_*(X; F), \quad \tau \neq 0 \quad (2-8)$$

for some angle $\psi \in \mathbf{R}$. Here $\tau = \tau(X, \eta, \xi; F)$ and $\det_F(c(\xi)) = e^{i\psi}$. Therefore, we obtain $\bar{\bar{\tau}} = e^{-i\psi} \bar{\tau} = \tau$. This proves that the anti-linear isomorphism (2-2) is involutive on the torsion τ and therefore it is involutive on any other element.

(B) obviously follows from (2-5) since the torsion τ does not depend on the Hermitian metric on F .

To prove (C) we observe that in the acyclic case the duality isomorphism D (cf. (1.3)) can be identified (after the canonical identification of the determinant lines $\det H_*(X; F)$ and $\det H_*(X; F^*)$ with \mathbf{C}) with the multiplication by

$$(-1)^{s\chi(X) \cdot \dim F \cdot (m+1)/2}.$$

The isomorphism $\psi : \det H_*(X; F^*) \rightarrow \det H_*(X; F)$ induced by (2-3) after these identifications coincides with the usual complex conjugation (as one sees from the definition of torsion). This implies our statement.

Now we will prove Theorem 2.2. If $\tau = e^{i\phi}\tau_0$, where τ_0 is real (with respect to the canonical involution (2-2)), then $\bar{\tau} = e^{-i\phi}\tau_0$ and from (2-5) we obtain $\bar{\tau}/\tau = e^{-2i\phi} = e^{-i \arg \det_F(c(\xi))}$. Therefore,

$$\mathbf{Ph}(\tau(X, \xi; F)) = \phi = \frac{1}{2} \arg \det_F(c(\xi)) \bmod \pi \mathbf{Z}.$$

□

§3. The Absolute Torsion

In this section we introduce a new concept of torsion which we call *absolute torsion*. It has some important advantages with respect to other similar notions of torsion: on one hand it is well defined and has no indeterminacy in most important cases including non-unimodular flat bundles. On the other hand it requires no additional topological information, such as Euler structures. We will show in the next section that the absolute torsion can be viewed as a natural high dimensional generalization of the Conway polynomial.

3.1. Basic assumption. In this section we will always deal with a closed *oriented* PL manifolds X of odd dimension m and a flat complex vector bundle $F \rightarrow X$, satisfying the following condition:

- (i) *The Stiefel-Whitney class $w_{m-1}(X) \in H^{m-1}(X, \mathbf{Z}_2)$ vanishes;*
- (ii) *The first Stiefel-Whitney class $w_1(F)$, viewed as a homomorphism $H_1(X; \mathbf{Z}) \rightarrow \mathbf{Z}_2$, vanishes on the 2-torsion subgroup of $H_1(X; \mathbf{Z})$.*

Note that condition (i) is automatically satisfied in the case $m \equiv 3 \pmod{4}$, as proven by W. Massey [Ma]. The condition (ii) holds for any orientable bundle F . Also, (ii) holds for any F assuming that $H_1(X)$ has no 2-torsion.

3.2. Canonical Euler structures. An Euler structure $\xi \in \text{Eul}(X)$ will be called *canonical* if it has trivial characteristic class $c(\xi) = 0 \in H_1(X)$. Since the mod 2 reduction of $c(\xi)$ coincides with $w_{m-1}(X)$, the assumption 3.1.(i) is necessary for the existence of canonical Euler structures. It is also sufficient: if $w_{m-1}(X) = 0$ then $c(\xi)$ admits a square root $c(\xi)^{1/2} \in H_1(X)$ and $\xi' = c(\xi)^{-1/2} \cdot \xi$ is a canonical Euler structure.

Since $c(h\xi) = h^2c(\xi)$, where $h \in H_1(X)$, we see that *the canonical Euler structure is unique if and only if the group $H_1(X)$ has no 2-torsion*.

In general, the number of canonical Euler structures on X equals the order of the 2-torsion subgroup of $H_1(X)$.

3.3. Canonical homological orientation. Recall that a *homological orientation* of X is a choice of an orientation of the line $\det H_*(X; \mathbf{R})$. It is observed in [T1] that if X is an oriented closed odd-dimensional manifold then there exists a *canonical homological orientation* of X , which sometimes depends on the choice of orientation of X . It is described as follows.

Fix an orientation on X . We assume that the dimension $m = \dim X$ is odd $m = 2r + 1$. For any $i \leq r$ fix an arbitrary basis $h_1^i, \dots, h_{b_i}^i \in H_i(X; \mathbf{R})$ and let $h_1^{m-i}, \dots, h_{b_i}^{m-i} \in H_{m-i}(X; \mathbf{R})$ be the dual basis, i.e.

$$\langle h_k^i \times h_l^{m-i}, [X] \rangle = \delta_{k,l}$$

where $[X] \in H_m(X \times X, X \times X - \Delta; \mathbf{R})$ is the class corresponding to the given orientation. Here $\Delta \subset X \times X$ denotes the diagonal. This gives volume forms $h^i = h_1^i \wedge \cdots \wedge h_{b_i}^i \in \det H_i(X; \mathbf{R})$ and $h^{m-i} = h_1^{m-i} \wedge \cdots \wedge h_{b_i}^{m-i} \in \det H_{m-i}(X; \mathbf{R})$ and hence we obtain a canonical nonzero element

$$h = h^0 \otimes (h^1)^{-1} \otimes h^2 \otimes \cdots \otimes (h^m)^{-1} \in \det H_*(X; \mathbf{R}).$$

Suppose that we reverse the orientation of X . Then the volume elements h^i with $i \leq r$ will be the same and each h^i with $i > r$ will be replaced by $(-1)^{b_i} h_i$. Hence we obtain that reversing the orientation of X changes h as follows: $h \mapsto (-1)^{s\chi(X)} \cdot h$. We arrive at the following:

3.4. Proposition. *Any closed oriented odd-dimensional manifold X has a canonical homological orientation. The canonical homological orientations of X does not depend on the orientation of X if and only if $s\chi(X)$ is even.*

3.5. Definition of absolute torsion. Our purpose in this section is to define a combinatorial invariant

$$\mathcal{T}(F) \in \det H_*(X; F) \tag{3-1}$$

for *arbitrary* flat \mathbf{C} -bundle F over a closed oriented odd-dimensional manifold X and a complex flat vector bundle $F \rightarrow X$, satisfying the conditions 3.1.

We emphasize that *we do not require F to be unimodular.*

We construct $\mathcal{T}(F)$ as follows. Choose a canonical Euler structure $\xi \in \text{Eul}(X)$ and consider the torsion

$$\mathcal{T}(F) = \tau(X, \eta, \xi; F) \in \det H_*(X; F), \tag{3-2}$$

where η is the canonical homological orientation, cf. 3.3, 3.4. The result *will not depend* on the choice of the canonical Euler structure ξ because of our assumption (ii) in 3.1. Indeed, replacing the canonical Euler structure ξ by another one, $h\xi$, where h belongs to the 2-torsion subgroup of $H_1(X)$, gives the following torsion $\tau(X, \eta, h\xi; F) = \det_F(h) \cdot \tau(X, \eta, \xi; F) = (-1)^{\langle w_1(F), h \rangle} \cdot \tau(X, \eta, \xi; F)$, and our statement now follows from condition (ii) in 3.1.

We call $\mathcal{T}(F)$ *the absolute torsion*.

An equivalent way to construct the absolute torsion consists in the following. Pick an arbitrary Euler structure $\xi \in \text{Eul}(X)$. Note that by our condition (i) in 3.1, *the characteristic class $c(\xi) \in H_1(X)$ is a square*. In fact, (in the additive notations) the mod 2 reduction of $c(\xi)$ vanishes, since it is Poincaré dual of the Stiefel-Whitney class w_{m-1} , which we assume to be zero. We know that there exists a square root $c(\xi)^{1/2} \in H_1(X)$. The indeterminacy in computing $c(\xi)^{1/2}$ can be described as follows $h \mapsto h \cdot c(\xi)^{1/2}$, where h belongs to 2-torsion of $H_1(X)$. By condition (ii) in (3.1) the monodromy of F along any loop representing the class $c(\xi)^{-1/2}$, is well defined and we may set

$$\mathcal{T}(F) = \det_F(c(\xi)^{-1/2}) \cdot \tau(X, \xi; F). \tag{3-3}$$

3.6. Conclusion. *Assume that X and F satisfy conditions 3.1. The absolute torsion $\mathcal{T}(F) \in \det H_*(X; F)$ is well defined. It is independent of the orientation of X under any of the following conditions:*

- (1) *If F is an even dimensional flat bundle;*
- (2) *If F is an odd dimensional flat bundle and the semi-characteristic $s\chi(X)$ is even.*

If F is an odd dimensional flat bundle and the semi-characteristic $s\chi(X)$ is of X odd, the absolute torsion changes the sign, when the orientation of X is reversed.

We now establish some properties of the absolute torsion.

3.7. Theorem (Duality). *Let F be a flat complex vector bundle over X and let F^* denote the dual flat vector bundle. Let*

$$D : \det H_*(X; F) \rightarrow \det H_*(X; F^*). \quad (3-4)$$

be the duality operator (1-12). Then,

$$D(\mathcal{T}(F)) = (-1)^{s\chi(X) \cdot \dim F \cdot (m+1)/2} \mathcal{T}(F^*), \quad (3-5)$$

where $m = \dim X$.

Proof. As we showed in section §2 (cf. (2-7)),

$$\begin{aligned} D(\tau(X, \eta, \xi; F)) = \\ (-1)^{s\chi(X) \cdot \dim F \cdot (m+1)/2} \cdot \det_F(c(\xi)) \cdot \tau(X, \eta, \xi; F^*) \in \det H_*(X; F^*). \end{aligned} \quad (3-6)$$

Dividing both sides by $\det_F(c(\xi)^{1/2})$ and observing that

$$\det_{F^*}(c(\xi)^{-1/2}) = \det_F(c(\xi)^{1/2})$$

we obtain

$$D\left(\frac{\tau(X, \eta, \xi; F)}{\det_F(c(\xi)^{1/2})}\right) = (-1)^{s\chi(X) \cdot \dim F \cdot (m+1)/2} \cdot \frac{\tau(X, \eta, \xi; F^*)}{\det_{F^*}(c(\xi)^{1/2})}$$

which, combined with the definition, proves out statement. \square

As the corollary we obtain that the absolute torsion is always real:

3.8. Theorem. *Assume that X and F satisfy conditions 3.1. Suppose that F admits a flat Hermitian metric. Recall the canonical involution on $\det H_*(X; F)$, cf. §2. Then $\mathcal{T}(F) \in \det H_*(X; F)$ is real:*

$$\overline{\mathcal{T}(F)} = \mathcal{T}(F). \quad (3-7)$$

Proof. Apply isomorphism (2-2), defined by a flat Hermitian metric on X , to both sides of (3-5). \square

§4. Absolute torsion and Conway polynomial

In this section we prove that the absolute torsion of a 3-dimensional manifold $X = X_K$, obtained by performing 0-surgery on a knot $K \subset S^3$, is precisely the *Conway polynomial* of K . This suggests to view the absolute torsion as a generalization of the Conway polynomial, which is applicable to high dimensions as well.

4.1. Recall that the Conway link polynomial is a function $L \mapsto \nabla_L$ from the set of isotopy classes of oriented links in S^3 into the ring of polynomials $\mathbf{Z}[z]$. This function is uniquely characterized by the following two properties:

- (i) its value on a trivial knot is equal to 1;
- (ii) for any skein triple of oriented links L_+, L_-, L_0 ,

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z). \quad (4-1)$$

Here by a skein triple of links we mean three oriented links coinciding outside a 3-ball and looking as the standard triple (positive crossing of 2 strands, negative crossing of 2 strands, two vertical strands) inside this ball.

Recall that the Conway polynomial $\nabla_K(z)$ of any oriented knot K involves only even powers of z .

4.2. Let $K \subset S^3$ be an oriented knot in S^3 and let the 3-manifold $X = X_K$ be obtained by 0-surgery on K . Then $H_1(X) = \mathbf{Z}$ (has no 2-torsion) and the semi-characteristic $s\chi(X) = 2$ is even. Also, the condition 3.1 is satisfied (since any orientable 3-manifold has a trivial tangent bundle). Hence, the absolute torsion $\mathcal{T}(F)$ is well defined for arbitrary flat bundle F over X (cf. §3). It is independent of the orientation of X .

We will consider *line* flat bundles F over X . It is clear that each such bundle is completely determined by the monodromy $a \in \mathbf{C}^*$ along a fixed generator $m \in H_1(X)$ (the meridian). We will denote this flat line bundle by F_a .

It is well known that the homology $H_*(X; F_a)$ is trivial if and only if $a \neq 1$ and a is not a root of the Alexander polynomial of K . This excludes finitely many points of \mathbf{C}^* . On the complement of these points the absolute torsion $\mathcal{T}(F_a)$ is a well defined \mathbf{C}^* -valued function of a . We shall compute this function in terms of the Conway polynomial of K .

4.3. Theorem. *Let $K \subset S^3$ be an oriented knot and $a \in \mathbf{C}^*$, $a \neq 1$, is not a root of the Alexander polynomial of K . Then the absolute torsion $\mathcal{T}(F_a)$ of the flat line bundle F_a over $X = X_K$ is given by*

$$\mathcal{T}(F_a) = \nabla_K(a^{1/2} - a^{-1/2}). \quad (4-2)$$

Proof. We begin by recalling the definition of $\nabla_K(z)$ in terms of torsions, given in [T1], section 4.3. Let Y be the exterior of K , i.e., the complement of an open regular neighborhood of K . We provide Y with a homology orientation defined by the basis $([pt], t)$ where $[pt]$ is the homology class of a point and t is the generator of $H_1(Y) = \mathbf{Z}$ represented by a meridian of K . (Note that $H_i(Y) = 0$ for $i \neq 0, 1$). Consider the (refined) Reidemeister torsion $\tau_0(Y)$ corresponding to the natural embedding of the ring $\mathbf{Z}[H_1(Y)] = \mathbf{Z}[t, t^{-1}]$ into its field of fractions $\mathbf{Q}(t)$. This torsion is an element of $\mathbf{Q}(t)$ defined up to multiplication by powers of t . Choose a representative $A(t) \in \mathbf{Q}(t)$ of $\tau_0(Y)$. It is known that $\overline{A(t)} = -t^m A(t)$ where $m \in \mathbf{Z}$ and the bar denotes the involution in $\mathbf{Q}(t)$ sending t to t^{-1} . Then

$$\nabla_K(t - t^{-1}) = -(t - t^{-1})^{-1} t^m A(t^2). \quad (4-3)$$

Note that here we use a normalization of ∇ different from the one in [T1]; this difference is responsible for the factor $(t - t^{-1})^{-1}$ on the right hand side. Note also that the product $(1 - t)A(t)$ is a polynomial in t representing the Alexander polynomial of K .

We can use the multiplicativity of torsions to compute the (refined) Reidemeister torsion $\tau_0(X)$ corresponding to the natural embedding of the ring $\mathbf{Z}[H_1(X)] = \mathbf{Z}[H_1(Y)] = \mathbf{Z}[t, t^{-1}]$ into its field of fractions $\mathbf{Q}(t)$. Since we are dealing with the sign-refined torsions we need to use the corresponding sign-refined multiplicativity theorem, [T1], Theorem 3.4.1. By this theorem, $\tau_0(X) = (-1)^\mu \tau_0(Y) \tau_0(X, Y)$ where μ is a certain residue modulo 2 and the pair (X, Y) is provided with a homology orientation induced by those in X and Y , as described in [T1], Theorem 3.4.1. A direct computation shows that in our setting $\mu = 0$ and $\tau_0(X, Y) = (1 - t)^{-1}$. Thus, $\tau_0(X) = (1 - t)^{-1} \tau_0(Y)$.

For any Euler structure ξ on X , we have a refined version $\tau_0(X, \xi) \in \mathbf{Q}(t)$ of $\tau_0(X)$. By duality, $\overline{\tau_0(X, \xi)} = c(\xi) \tau_0(X, \xi)$, see [T3], section 2.7. We take ξ to be the canonical Euler structure on X , so that $\overline{\tau_0(X, \xi)} = \tau_0(X, \xi)$. By the argument above, the product $A(t) = (1 - t) \tau_0(X, \xi)$ is a representative of $\tau_0(Y)$. It is clear that $\overline{A(t)} = -t^{-1} A(t)$ so that

$$\nabla_K(t - t^{-1}) = -(t - t^{-1})^{-1} t^{-1} A(t^2) = \tau_0(X, \xi)(t^2)$$

where $\tau_0(X, \xi)(t^2)$ is obtained from the rational function $\tau_0(X, \xi) = \tau_0(X, \xi)(t) \in \mathbf{Q}(t)$ by doubling all powers of t . It follows from definitions that for any non-zero complex number a , the number $\mathcal{T}(F_a)$ is obtained from the rational function $\tau_0(X, \xi) \in \mathbf{Q}(t)$ by the substitution $t = a$. This implies the claim of the theorem. \square

4.4. Remark. There exist analogues of Theorem 4.3 for links, which involve the one-variable and multi-variable Conway polynomials.

Let us briefly describe the case of one-variable Conway polynomial.

Let $L = \{\ell_1, \ell_2, \dots, \ell_\mu \subset S^3\}$ be an oriented link. We will assume that the Conway polynomial of L is nonzero. Let $\lambda_{i,j}$ denote the linking number of i -th and j -th components, $i \neq j$. Set $\lambda_j = \sum \lambda_{i,j}$, where the summation is taken with respect to $i = 1, \dots, \mu$, $i \neq j$. Consider the closed 3-manifold $X = X_L$ obtained from S^3 by a surgery along the link L , with framing $-\lambda_j$ along the component ℓ_j , for each $j = 1, 2, \dots, \mu$. There is a canonical homomorphism $H_1(X) \rightarrow \mathbf{Z}$ (determined by the Seifert surface of L) and so, as above, for any complex number $a \in \mathbf{C}^*$ we have a canonical flat line bundle F_a over X . If $a \neq 1$ is not a root of the Alexander polynomial, then $H_*(X; F_a) = 0$, and we obtain $\mathcal{T}(F_a) \in \mathbf{C}^*$. In this situation the following formula holds

$$\mathcal{T}(F_a) = \pm \frac{\nabla_L(a^{1/2} - a^{-1/2})}{(a^{1/2} - a^{-1/2})^{\mu-1}}. \quad (4-4)$$

It expresses the absolute torsion $\mathcal{T}(F_a)$ in terms of the Conway polynomial $\nabla_L(z)$ of L . Formula (4-4) allows also to find the Conway polynomial $\nabla_L(z)$ knowing $\mathcal{T}(F_a)$. To see this we recall that the Conway polynomial $\nabla_L(z)$ involves only powers z^k with k being of the same parity as $\mu - 1$; hence $\nabla_L(z)/z^{\mu-1}$ is a Laurent polynomial in z^2 .

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